

CHARACTER DEGREES OF EXTENSIONS OF THE SUZUKI GROUPS ${}^2B_2(q^2)$

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ABSTRACT. Let S be a Suzuki group ${}^2B_2(q^2)$, where $q^2 = 2^{2f+1}$, $f \geq 1$. In this paper, we determine the degrees of the ordinary complex irreducible characters of every group G such that $S \leq G \leq \text{Aut}(S)$.

1. INTRODUCTION

Given a finite group G , let $\text{cd}(G) = \{\chi(1) \mid \chi \in \text{Irr}(G)\}$ be the set of degrees of the ordinary complex irreducible characters of G . A common approach in the studying of nonsolvable groups with a given property on irreducible character degrees begins by examining the property on simple and almost simple groups. Among these, depending on the given property, the most work is done generally on simple groups S with few character degrees and even more so, on groups G with $S \leq G \leq \text{Aut}(S)$ such that a few characters of S are extendible to G , (see [3, 5, 6]).

The 2-dimensional projective special linear groups $\text{PSL}_2(q)$, $q > 3$ with 4 or 5 degrees and the Suzuki groups ${}^2B_2(q^2)$ with 6 degrees are two families of nonabelian simple groups which have the least number of character degrees. In addition, if S is a simple group of Lie type, then by Theorems 2.4 and 2.5 of [7], all unipotent characters of S apart from a few cases extend to $\text{Aut}(S)$. By using [1, Chapter 13] and [7] for simple groups of Lie type and the Atlas [2] and hook-partitions for sporadic and alternating groups, it follows that the families $\text{PSL}_2(q)$ and ${}^2B_2(q^2)$ are among the nonabelian simple groups with a small number of degrees extendible to $\text{Aut}(S)$. All of these emphasize to have an explicit result on the set of degrees of the irreducible characters of every almost simple group G for which $S = \text{PSL}_2(q)$ or ${}^2B_2(q^2)$. For $S = \text{PSL}_2(q)$, this has been done in [9]. In this paper, we consider the Suzuki groups ${}^2B_2(q^2)$, where $q^2 = 2^{2f+1}$, $f \geq 1$.

We will use the notation of [8] for the Suzuki groups, in which instead of q we write q^2 . By Theorem 5 of [8], the character degree set of ${}^2B_2(q^2)$ is

$$\text{cd}({}^2B_2(q^2)) = \{1, q^4, q^4 + 1, (q^2 - r + 1)(q^2 - 1), (q^2 + r + 1)(q^2 - 1), r(q^2 - 1)/2\},$$

where $r = 2^{f+1}$.

Our main result is the following:

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Theorem A. *Let $S = {}^2B_2(q^2)$, where $q^2 = 2^{2f+1}$, $f \geq 1$ and let $S \leq G \leq \text{Aut}(S)$ with $|G : S| = d$. Then the set of irreducible character degrees of G is*

$$\text{cd}(G) = \{1, q^4, r(q^2-1)/2\} \cup \{(q^4+1)a, (q^2-r+1)(q^2-1)b, (q^2+r+1)(q^2-1)c : a, b, c \mid d\},$$

with the following exceptions:

- (i) *If $G = \text{Aut}(S)$, then $a \neq 1$,*
- (ii) *If $f \equiv 1$ or $2 \pmod{4}$, and $G = \text{Aut}(S)$, then $b \neq 1$ and $c \neq 3$,*
- (iii) *If $f \equiv 0$ or $3 \pmod{4}$, and $G = \text{Aut}(S)$, then $b \neq 3$ and $c \neq 1$.*

As an immediate consequence of Theorem A, we obtain the following.

Corollary B. *Let $S = {}^2B_2(q^2)$, where $q^2 = 2^{2f+1}$, $f > 1$ and let $S < G \leq \text{Aut}(S)$ with $|G : S| = d$. Then $|\text{cd}(G)| \geq 7$, and if d is not a prime, then $|\text{cd}(G)| \geq 9$.*

For a normal subgroup S of a finite group G , we denote by $\text{cd}(G \mid S)$, the set of degrees of irreducible characters of G whose kernels do not contain S . Observe that Corollary B can be used to improve the conclusions of [3], on finite groups G with a nonsolvable normal subgroup S such that $|\text{cd}(G \mid S)| \leq 5$. See [3, Theorems A, B and Corollary C].

At the end of this section, we sketch our proof of Theorem A as follows. First, note that $S \trianglelefteq G$, so

$$\text{cd}(G) = \bigcup_{\theta \in \text{Irr}(S)} \text{cd}(G \mid \theta),$$

in which $\text{cd}(G \mid \theta) = \{\chi(1) \mid \chi \in \text{Irr}(G), [\chi_S, \theta] \neq 0\}$. Also, the group of outer automorphisms of S is cyclic. Thus by [4, Corollary 11.22], every $\theta \in \text{Irr}(S)$ extends to its stabilizer $I_G(\theta)$ in G and so Gallagher's theorem [4, Corollary 6.17] yields that $\text{cd}(I_G(\theta) \mid \theta) = \{\theta(1)\}$. Using Clifford's theorem [4, Theorem 6.11], we obtain $\text{cd}(G \mid \theta) = \{|G : I_G(\theta)|\theta(1)\}$. So what is important here, is to find the stabilizer of any $\theta \in \text{Irr}(S)$ in G . To do this we need several number theoretic results on divisors of $q^4 + 1$ and $q^2 \pm r + 1$, which will be provided in section 2.

2. PRELIMINARIES

We first establish some notation which will remain fixed throughout this paper. Simultaneously, we mention some facts from [8] about the Suzuki groups which will be needed in this paper. Let f be a positive integer, let $q^2 = 2^{2f+1}$ and $r = 2^{f+1}$. We will denote the Suzuki group ${}^2B_2(q^2)$ by S . Note that S is a simple group of order $(q^4 + 1)q^4(q^2 - 1)$ and its order is not divisible by 3. We may regard S as a subgroup of $\text{GL}_4(q^2)$. Using the notation of [8], we know that S has $q^2 + 3$ conjugacy classes of elements, consisting of a class of involutions with a representative σ , two classes of elements of order 4 with representatives ρ, ρ^{-1} and q^2 classes of semisimple elements. Also, by [8, Theorem 4], any semisimple element of S is conjugate to an element of the cyclic subgroups A_0, A_1, A_2 of S .

In order to determine which subgroups of $\text{Aut}(S)$ appear as stabilizer of irreducible characters of S , we first consider the action of outer automorphisms of S on conjugacy

classes. According to [8, Theorem 11], the group of outer automorphisms of S is cyclic of odd order $2f + 1$ and is generated by a field automorphism. To be more precise, let $\mathcal{G} = \text{Gal}(\mathbb{F}_{q^2}/\mathbb{F}_2)$. Note that \mathcal{G} is generated by the Frobenius automorphism $\overline{\varphi}$, where $\overline{\varphi}(\zeta) = \zeta^2$ for all $\zeta \in \mathcal{G}$. Then $\text{Out}(S)$ is generated by the outer automorphism φ of S induced by $\overline{\varphi}$. By [8], we obtain the following lemma on the action of φ on conjugacy classes of S .

Lemma 2.1. *Assume notation as above and let n be an integer. The automorphism φ fixes the conjugacy classes of σ , ρ , ρ^{-1} in S . Also, if x is a semisimple element of S , then φ sends the conjugacy class of x to the class of x^2 , and φ^n sends the class of x to the class of x^{2^n} .*

We now consider the action of the field automorphism φ on the irreducible characters of S . By [8, Section 17], the nonlinear irreducible characters of S along with their degrees and the number of characters of each degree are as in the following table:

X	q^4	1
X_i	$q^4 + 1$	$q^2/2 - 1$
Y_j	$(q^2 - r + 1)(q^2 - 1)$	$(q^2 + r)/4$
Z_k	$(q^2 + r + 1)(q^2 - 1)$	$(q^2 - r)/4$
W_l	$r(q^2 - 1)/2$	2

The characters of S of degrees 1 and q^4 are unique, so they are invariant under φ . Also, the characters W_l with $l = 1, 2$ are equal on all classes of semisimple elements, (see [8, Theorem 13]). So they are invariant under φ by Lemma 2.1. Thus by [4, Corollaries 11.22 and 6.17], we obtain the following result.

Lemma 2.2. *All characters of S of degrees 1, q^4 and $r(q^2 - 1)/2$ are invariant under φ and each extends to $2f + 1$ distinct irreducible characters of $\text{Aut}(S)$.*

It remains to determine the subgroups of $\text{Aut}(S)$ that occur as stabilizers of characters X_i , Y_j and Z_k of S . For this, we collect in this section, several number theoretic lemmas, which will be used frequently in our proofs.

Lemma 2.3. [9, Lemma 4.9]) *If p is a prime and m, n are positive integers such that $m \mid n$, then*

- (i) $(p^n - 1, p^m - 1) = p^m - 1$,
- (ii) $(p^n - 1, p^m + 1) = \begin{cases} (p - 1, 2) & \text{if } n/m \text{ is odd,} \\ p^m + 1 & \text{if } n/m \text{ is even,} \end{cases}$
- (iii) $(p^n + 1, p^m - 1) = (p - 1, 2)$,
- (iv) $(p^n + 1, p^m + 1) = \begin{cases} p^m + 1 & \text{if } n/m \text{ is odd,} \\ (p - 1, 2) & \text{if } n/m \text{ is even.} \end{cases}$

Lemma 2.4. *If n is a proper positive divisor of $2f + 1$, then*

- (i) $(q^4 + 1, 2^n \pm 1) = 1$,

$$(ii) \ (q^4 + 1, q^2 - 2^n) = \begin{cases} 2^{2n} + 1 & \text{if } 2f + 1 \equiv n \pmod{4}, \\ 1 & \text{otherwise,} \end{cases}$$

$$(iii) \ (q^4 + 1, q^2 + 2^n) = \begin{cases} 2^{2n} + 1 & \text{if } 2f + 1 \equiv -n \pmod{4}, \\ 1 & \text{otherwise.} \end{cases}$$

Proof. Part (i) follows from Lemma 2.3. Let $d = (q^4 + 1, q^2 - 2^n)$. Since d is odd, we have $d \mid 2^{2f-n+1} - 1$ and hence $d \mid 2^{4f-2n+2} - 1$. Also $d \mid 2^{4f+2} + 1$, thus $d \mid 2^{4f-2n+2}(2^{2n} + 1)$ and so $d \mid 2^{2n} + 1$. On the other hand, by Lemma 2.3, we have $2^{2n} + 1 \mid q^4 + 1$ and

$$(q^2 - 2^n, 2^{2n} + 1) = (2^{2f-n+1} - 1, 2^{2n} + 1) = \begin{cases} 2^{2n} + 1 & \text{if } (2f - n + 1)/2n \text{ is even,} \\ 1 & \text{otherwise.} \end{cases}$$

Thus, if $2f + 1 \equiv n \pmod{4}$, then $2^{2n} + 1 \mid q^2 - 2^n$ and so $d = 2^{2n} + 1$, while if $2f + 1 \not\equiv n \pmod{4}$, then $(q^2 - 2^n, 2^{2n} + 1) = 1$ and $d = 1$. Hence (ii) follows. A similar argument applies to (iii). Now the proof is complete. \square

In the following lemma, we shall use $p^n \nmid d$ to denote that p^n is the highest power of prime number p which divides d .

Lemma 2.5. *Let n be a proper positive divisor of $2f + 1$. Then*

$$(i) \ (q^2 \pm r + 1, q^2 - 2^n) = \begin{cases} 2^n \pm 2^{(n+1)/2} + 1 & \text{if } 8 \mid 2f - n + 1, \\ 2^n \mp 2^{(n+1)/2} + 1 & \text{if } 4 \nmid 2f - n + 1, \\ 1 & \text{otherwise,} \end{cases}$$

$$(ii) \ (q^2 \pm r + 1, q^2 + 2^n) = \begin{cases} 2^n \pm 2^{(n+1)/2} + 1 & \text{if } 8 \mid 2f + n + 1, \\ 2^n \mp 2^{(n+1)/2} + 1 & \text{if } 4 \nmid 2f + n + 1, \\ 1 & \text{otherwise.} \end{cases}$$

Proof. (i) Observe first that

$$q^4 + 1 = (q^2 - r + 1)(q^2 + r + 1).$$

If $2f + 1 \not\equiv n \pmod{4}$, then Lemma 2.4(ii) yields $(q^2 \pm r + 1, q^2 - 2^n) = 1$. So we may assume that $2f + 1 \equiv n \pmod{4}$. If $2f + 1 \equiv n \pmod{8}$, then $2^{2n} + 1 \mid 2^{(2f-n+1)/2} - 1$ by Lemma 2.3(ii) and hence

$$(2^{(2f-n+1)/2} - 1)(2^{(2f+n+1)/2} - 1) \equiv 0 \pmod{2^{2n} + 1}.$$

Thus

$$q^2 \pm r + 1 \equiv 2^{(2f-n+1)/2}(2^n \pm 2^{(n+1)/2} + 1) \pmod{2^{2n} + 1}.$$

Since $2^{2n} + 1 = (2^n - 2^{(n+1)/2} + 1)(2^n + 2^{(n+1)/2} + 1)$, we have

$$2^n + 2^{(n+1)/2} + 1 \mid q^2 + r + 1 \quad \text{and} \quad 2^n - 2^{(n+1)/2} + 1 \mid q^2 - r + 1.$$

Now, Lemma 2.4(ii) applies to yield that

$$(q^2 \pm r + 1, q^2 - 2^n) = 2^n \pm 2^{(n+1)/2} + 1.$$

Finally, if $4 \nmid 2f - n + 1$, then $2^{2n} + 1 \mid 2^{(2f-n+1)/2} + 1$ by Lemma 2.3(iv). Thus

$$(2^{(2f-n+1)/2} + 1)(2^{(2f+n+1)/2} + 1) \equiv 0 \pmod{2^{2n} + 1}.$$

Therefore,

$$q^2 \pm r + 1 \equiv -2^{(2f-n+1)/2}(2^n \mp 2^{(n+1)/2} + 1) \pmod{2^{2n} + 1}.$$

Again, because of $2^{2n} + 1 = (2^n - 2^{(n+1)/2} + 1)(2^n + 2^{(n+1)/2} + 1)$, we deduce from Lemma 2.4(ii) that

$$(q^2 \pm r + 1, q^2 - 2^n) = 2^n \mp 2^{(n+1)/2} + 1.$$

Hence (i) follows.

(ii) Note that if $8 \mid 2f + n + 1$, then $2^{2n} + 1 \mid 2^{(2f+n+1)/2} - 1$ by Lemma 2.3. Also, if $4 \nmid 2f + n + 1$, then $2^{2n} + 1 \mid 2^{(2f+n+1)/2} + 1$, so we can apply the above argument for (ii). The proof is now complete. \square

Lemma 2.6. *Let m, n be distinct proper positive divisors of $2f + 1$ with $m \mid n$. Set*

$$d_1 = (q^2 + r + 1, q^2 - 2^n) \quad \text{or} \quad (q^2 + r + 1, q^2 + 2^n),$$

$$d_2 = (q^2 + r + 1, q^2 - 2^m) \quad \text{or} \quad (q^2 + r + 1, q^2 + 2^m).$$

If $d_1 > 1$, then $d_1 \neq d_2$ except when $m = 1, n = 3$ and one of the following holds:

- (i) $f \equiv 0 \pmod{4}$, $d_1 = (q^2 + r + 1, q^2 + 8)$, $d_2 = (q^2 + r + 1, q^2 - 2)$,
- (ii) $f \equiv 3 \pmod{4}$, $d_1 = (q^2 + r + 1, q^2 - 8)$, $d_2 = (q^2 + r + 1, q^2 + 2)$.

Similarly, replacing $q^2 + r + 1$ above, by $q^2 - r + 1$ yields that $d_1 \neq d_2$ except when $m = 1, n = 3$ and one of the following holds:

- (iii) $f \equiv 1 \pmod{4}$, $d_1 = (q^2 - r + 1, q^2 - 8)$, $d_2 = (q^2 - r + 1, q^2 + 2)$,
- (iv) $f \equiv 3 \pmod{4}$, $d_1 = (q^2 - r + 1, q^2 + 2)$, $d_2 = (q^2 - r + 1, q^2 - 2)$.

In cases (i)-(iv), we have $d_1 = d_2 = 5$.

Proof. Note that exactly one of $2f \pm n + 1$ is divisible by 4. So only one of $(q^2 + r + 1, q^2 \pm 2^n)$ is greater than 1 by Lemma 2.5. The same is true for $(q^2 - r + 1, q^2 \pm 2^n)$. Also, if $n > 3$, then clearly $d_1 > 5$, while if $n = 3$, then $d_1 = 5$ or 13. By comparing the values of d_1 and d_2 , that is the numbers $2^n \pm 2^{(n+1)/2} + 1$ with $1, 2^m \pm 2^{(m+1)/2} + 1$, it is relevant to see that $d_1 \neq d_2$ with the exception that $m = 1, n = 3$ and $d_1 = d_2 = 5$. By Lemma 2.5, $d_1 = d_2 = 5$ if and only if one of (i)-(iv), given above holds. \square

The following lemma will be useful when working with characters Y_j, Z_k of S .

Lemma 2.7. *Let ε be a complex n th root of unity and let i, j, k be integers such that $k^2 \equiv -1 \pmod{n}$. Then*

$$(1) \quad \varepsilon^{il} + \varepsilon^{-il} + \varepsilon^{ilk} + \varepsilon^{-ilk} = \varepsilon^{jl} + \varepsilon^{-jl} + \varepsilon^{jlk} + \varepsilon^{-jlk}, \quad \text{where } l = 1, k - 1,$$

if and only if one of the congruents $i \equiv \pm j \pmod{n}$ or $i \equiv \pm jk \pmod{n}$ holds.

Proof. Suppose that (1) holds for $l = 1, k - 1$. Observe that

$$\begin{aligned} &(\varepsilon^i + \varepsilon^{-i} - \varepsilon^j - \varepsilon^{-j})(\varepsilon^i + \varepsilon^{-i} - \varepsilon^{jk} - \varepsilon^{-jk}) = 2 + \varepsilon^{2i} + \varepsilon^{-2i} \\ &\quad - (\varepsilon^i + \varepsilon^{-i})(\varepsilon^j + \varepsilon^{-j} + \varepsilon^{jk} + \varepsilon^{-jk}) + \varepsilon^{j(k-1)} + \varepsilon^{-j(k-1)} + \varepsilon^{j(k+1)} + \varepsilon^{-j(k+1)}. \end{aligned}$$

Replacing $\varepsilon^j + \varepsilon^{-j} + \varepsilon^{jk} + \varepsilon^{-jk}$ above, by $\varepsilon^i + \varepsilon^{-i} + \varepsilon^{ik} + \varepsilon^{-ik}$ yields

$$\begin{aligned} & (\varepsilon^i + \varepsilon^{-i} - \varepsilon^j - \varepsilon^{-j})(\varepsilon^i + \varepsilon^{-i} - \varepsilon^{jk} - \varepsilon^{-jk}) \\ &= -\varepsilon^{i(k-1)} - \varepsilon^{-i(k-1)} - \varepsilon^{i(k+1)} - \varepsilon^{-i(k+1)} + \varepsilon^{j(k-1)} + \varepsilon^{-j(k-1)} + \varepsilon^{j(k+1)} + \varepsilon^{-j(k+1)} \\ &= 0, \end{aligned}$$

where the last equality follows from (1) with $l = k - 1$, since $k^2 \equiv -1 \pmod{n}$. Therefore,

$$\varepsilon^i + \varepsilon^{-i} = \varepsilon^j + \varepsilon^{-j} \quad \text{or} \quad \varepsilon^i + \varepsilon^{-i} = \varepsilon^{jk} + \varepsilon^{-jk}.$$

Now, Lemma 4.17 of [9] yields that $i \equiv \pm j \pmod{n}$ or $i \equiv \pm jk \pmod{n}$. Conversely, since $k^2 \equiv -1 \pmod{n}$, it is clear that each of the above congruents yields (1) for all integers l . \square

3. STABILIZERS OF CHARACTERS OF DEGREES $q^4 + 1$ AND $q^2 \pm r + 1$

In this section, we shall determine the subgroups of $\text{Aut}(S)$ that occur as stabilizers of characters X_i, Y_j, Z_k of S . According to [8, Theorem 9], the group S contains three cyclic subgroups A_0, A_1 and A_2 of order $q^2 - 1, q^2 + r + 1$ and $q^2 - r + 1$, respectively. Also, by [8, Theorem 13], the characters X_i are equal everywhere except on $A_0 - \{1\}$. The same property is valid for characters Y_j and Z_k in corresponding with subgroups A_1 and A_2 , respectively.

First we consider the characters X_i , where $1 \leq i \leq q^2/2 - 1$. Let ε_0 be a primitive $(q^2 - 1)$ th root of unity. If ξ_0 is a generator of A_0 , then

$$X_i(\xi_0^l) = \varepsilon_0^{il} + \varepsilon_0^{-il},$$

for all $l, 1 \leq l \leq q^2 - 1$. By Lemma 2.1, the action of a power φ^n of φ on ξ_0^l is as follows:

$$X_i \varphi^n(\xi_0^l) = X_i((\xi_0^l)^{\varphi^{-n}}) = X_i(\xi_0^{-l2^n}) = \varepsilon_0^{il2^n} + \varepsilon_0^{-il2^n}.$$

Lemma 3.1. *Let i, n be positive integers such that $1 \leq i \leq q^2/2 - 1$ and $n \mid 2f + 1$.*

- (i) *The character X_i is invariant under φ^n if and only if $q^2 - 1 \mid (2^n - 1)i$.*
- (ii) *Let $N = \langle \varphi^n \rangle$ and set $i = (q^2 - 1)/(2^n - 1)$. Then N is the stabilizer of X_i in $\langle \varphi \rangle$ unless when $n = 1$, in which case N does not stabilize any X_i .*

Proof. (i) By Lemma 2.1 and the above discussion, we know that X_i is invariant under φ^n if and only if $X_i(\xi_0^l) = X_i^{\varphi^n}(\xi_0^l)$, for all $l, 1 \leq l \leq q^2 - 1$, hence if and only if

$$\varepsilon_0^{il} + \varepsilon_0^{-il} = \varepsilon_0^{il2^n} + \varepsilon_0^{-il2^n},$$

for all l . In particular, by taking $l = 1$, we obtain $i2^n \equiv \pm i \pmod{q^2 - 1}$, see [9, Lemma 4.7]. Note that the congruent $i2^n \equiv -i \pmod{q^2 - 1}$ is impossible, otherwise Lemma 2.3(ii) yields that $q^2 - 1 \mid i$, a contradiction to $1 \leq i \leq q^2/2 - 1$. Hence X_i is invariant under φ^n if and only if $q^2 - 1 \mid (2^n - 1)i$, as claimed.

(ii) Note that if $n = 1$, then part (i) implies there is no X_i stabilized by $N = \langle \varphi \rangle$. So we may assume that $n > 1$. It follows that

$$i = \frac{q^2 - 1}{2^n - 1} \leq \frac{q^2 - 1}{3} < \frac{q^2}{2} - 1.$$

By Lemma 2.3, i is an integer, hence $X_i \in \text{Irr}(S)$. Moreover, we have $q^2 - 1 = (2^n - 1)i$, so by part (i), X_i is stabilized by N . If the stabilizer, T , of X_i in $\langle \varphi \rangle$ properly contains N , then $T = \langle \varphi^t \rangle$ for some divisor t of n with $1 \leq t < n$. By part (i), we have $q^2 - 1 \mid (2^t - 1)i$. It follows that $2^n - 1 \mid 2^t - 1$, a contradiction, since $t < n$. Therefore, N is the stabilizer of X_i in $\langle \varphi \rangle$. \square

We next consider the characters Y_j , where $1 \leq j \leq (q^2 + r)/4$. Let ε_1 be a primitive $(q^2 + r + 1)$ th root of unity. If ξ_1 is a generator of A_1 , then

$$Y_j(\xi_1^l) = -(\varepsilon_1^{jl} + \varepsilon_1^{-jl} + \varepsilon_1^{jlq^2} + \varepsilon_1^{-jlq^2}),$$

for all l , $1 \leq l \leq q^2 + r + 1$. Note that by [8, Theorem 13], the characters Y_j are equal everywhere except on $A_1 - \{1\}$.

Lemma 3.2. *Let j, n be positive integers such that $1 \leq j \leq (q^2 + r)/2$ and $n \mid 2f + 1$. Then the character Y_j is invariant under φ^n if and only if*

$$q^2 + r + 1 \mid (q^2 - 2^n)j \quad \text{or} \quad q^2 + r + 1 \mid (q^2 + 2^n)j.$$

Proof. The character Y_j is invariant under φ^n if and only if

$$Y_j(\xi_1^l) = Y_j^{\varphi^n}(\xi_1^l) = Y_j(\xi_1^{-l2^n}),$$

for all l , $1 \leq l \leq q^2 + r + 1$, hence if and only if

$$\varepsilon_1^{jl} + \varepsilon_1^{-jl} + \varepsilon_1^{jlq^2} + \varepsilon_1^{-jlq^2} = \varepsilon_1^{jl2^n} + \varepsilon_1^{-jl2^n} + \varepsilon_1^{jlq^2 2^n} + \varepsilon_1^{-jlq^2 2^n},$$

for all l . Since $q^4 \equiv -1 \pmod{q^2 + r + 1}$, so by Lemma 2.7, this holds if and only if

$$j2^n \equiv \pm j \pmod{q^2 + r + 1} \quad \text{or} \quad jq^2 \equiv \pm j2^n \pmod{q^2 + r + 1}.$$

The congruents $j2^n \equiv \pm j \pmod{q^2 + r + 1}$ are impossible, otherwise Lemma 2.4(i) yields $q^2 + r + 1 \mid j$, a contradiction to $1 \leq j \leq (q^2 + r)/4$. Hence the result follows. \square

Lemma 3.3. *Let n be a positive divisor of $2f + 1$. Set $N = \langle \varphi^n \rangle$. Then N is the stabilizer in $\langle \varphi \rangle$ of some Y_j unless one of the following cases occur.*

- (i) $n = 1$ and $f \equiv 1$ or $2 \pmod{4}$,
- (ii) $n = 3$ and $f \equiv 0$ or $3 \pmod{4}$.

In cases (i) and (ii), N does not stabilize any Y_j .

Proof. If Y_1 is invariant under φ^n , then $q^2 + r + 1 \mid q^2 - 2^n$ or $q^2 + r + 1 \mid q^2 + 2^n$ by Lemma 3.2. (It is relevant to see that if $n < 2f + 1$, then none of the numbers $q^2 \pm r + 1$ divide $q^2 - 2^n$ or $q^2 + 2^n$.) Hence we must have $n = 2f + 1$. It follows that $N = S$ is the stabilizer of Y_1 in $\langle \varphi \rangle$.

Assume then that $n < 2f + 1$. If $n = 1$ and $f \equiv 1$ or $2 \pmod{4}$, then Lemma 2.5 yields $(q^2 + r + 1, q^2 \pm 2) = 1$. In this case, we know by Lemma 3.2 that $N = \langle \varphi \rangle$ does

not stabilize any Y_j . Assume now that $n = 3$ and $f \equiv 0$ or $3 \pmod{4}$. If $N = \langle \varphi^3 \rangle$ stabilizes some Y_j , then Lemma 3.2 yields that

$$q^2 + r + 1 \mid (q^2 - 8)j \quad \text{or} \quad q^2 + r + 1 \mid (q^2 + 8)j.$$

By Lemma 2.6, we have

$$(q^2 + r + 1, q^2 - 8) = (q^2 + r + 1, q^2 + 2) = 5 \quad \text{or} \quad (q^2 + r + 1, q^2 + 8) = (q^2 + r + 1, q^2 - 2) = 5,$$

hence

$$q^2 + r + 1 \mid (q^2 + 2)j \quad \text{or} \quad q^2 + r + 1 \mid (q^2 - 2)j.$$

Thus $\langle \varphi \rangle$ stabilizes Y_j . Since $n = 3$ is a proper divisor of $2f + 1$, so $N < \langle \varphi \rangle$. It follows that N is not the stabilizer of any Y_j . In particular, note that if (ii) occurs, then $\langle \varphi \rangle$ is the stabilizer of Y_j with $j = (q^2 + r + 1)/5$.

We may therefore assume that exceptions (i) and (ii) does not occur. Note that only one of $2f \pm n + 1$ is divisible by 4. So exactly one of $(q^2 + r + 1, q^2 \pm 2^n)$ is greater than 1 and in fact ≥ 5 by Lemma 2.5. We may assume without loss that $(q^2 + r + 1, q^2 - 2^n) > 1$. The same argument holds if we replace $q^2 - 2^n$ by $q^2 + 2^n$. Let j be the least positive integer such that N is contained in the stabilizer of Y_j in $\langle \varphi \rangle$. Then by Lemma 3.2,

$$j = \frac{q^2 + r + 1}{(q^2 + r + 1, q^2 - 2^n)}.$$

Since

$$j \leq \frac{q^2 + r + 1}{5} < \frac{q^2 + r}{4},$$

it follows that $Y_j \in \text{Irr}(S)$. If the stabilizer, T , of Y_j in $\langle \varphi \rangle$ properly contains N , then $T = \langle \varphi^t \rangle$ for some divisor t of n with $1 \leq t < n$. By Lemma 3.2, we have

$$q^2 + r + 1 \mid (q^2 - 2^t)j \quad \text{or} \quad q^2 + r + 1 \mid (q^2 + 2^t)j.$$

It follows that one of $(q^2 + r + 1, q^2 \pm 2^t)$ is greater than 1. Again, we may assume that $(q^2 + r + 1, q^2 + 2^t) > 1$. Then $(q^2 + 2^t)/(q^2 + r + 1, q^2 - 2^n)$ is an integer and so $(q^2 + r + 1, q^2 - 2^n) \mid q^2 + 2^t$. Thus

$$j \geq \frac{q^2 + r + 1}{(q^2 + r + 1, q^2 + 2^t)}.$$

For equality, let $k = (q^2 + r + 1)/(q^2 + r + 1, q^2 + 2^t)$. Then T and hence N is contained in the stabilizer of Y_k in $\langle \varphi \rangle$. By the choice of j , we must have $j = k$, but this is impossible by Lemma 2.6. Hence $N = T$ and the proof is complete. \square

Finally we consider the characters Z_k , where $1 \leq j \leq (q^2 - r)/4$. Let ε_2 be a primitive $(q^2 - r + 1)$ th root of unity. If ξ_2 is a generator of A_2 , then

$$Z_k(\xi_2^l) = -(\varepsilon_2^{jl} + \varepsilon_2^{-jl} + \varepsilon_2^{jlq^2} + \varepsilon_2^{-jlq^2}).$$

Note that by [8, Theorem 13], the characters Z_k are equal everywhere except on $A_2 - \{1\}$. Reasoning as in the proof of Lemmas 3.2 and 3.3, we obtain the following result on the stabilizers of characters Z_k .

Lemma 3.4. *Let n be a positive divisor of $2f + 1$. Set $N = \langle \varphi^n \rangle$. Then N is the stabilizer in $\langle \varphi \rangle$ of some Z_k unless one of the following cases occur.*

- (i) $n = 1$ and $f \equiv 0$ or $3 \pmod{4}$,
- (ii) $n = 3$ and $f \equiv 1$ or $2 \pmod{4}$.

In cases (i) and (ii), N does not stabilize any Z_k .

4. MAIN RESULTS

In this section, we complete the proof of Theorem A. It suffices by Lemma 2.2 to find the degrees of irreducible characters of G with $S \leq G \leq \text{Aut}(S)$ lying over characters X_i , Y_j and Z_k of S .

Theorem 4.1. *Let $S = {}^2B_2(q^2)$, where $q^2 = 2^{2f+1}$, $f \geq 1$ and let $S \leq G \leq \text{Aut}(S)$ with $|G : S| = d$. Then the degrees of the irreducible characters of G lying over characters of S of degree $q^4 + 1$ are precisely $(q^4 + 1)a$, where a is a positive divisor of d , with the exception that $a \neq 1$ when $G = \text{Aut}(S)$.*

Proof. We have $G = S\langle\varphi^{(2f+1)/d}\rangle$. As it was mentioned in section 1, if $\theta \in \text{Irr}(S)$, then $\text{cd}(G \mid \theta) = \{|G : I_G(\theta)|\theta(1)\}$. Thus, for $a \mid d$, there is a character of G of degree $(q^4 + 1)a$ lying over a character θ of S of degree $q^4 + 1$ if and only if

$$I_G(\theta) = S\langle\varphi^n\rangle, \quad \text{where } n = \left(\frac{2f+1}{d}\right)a.$$

By Lemma 3.1, such a character θ of S exists if and only if $n \neq 1$. Since $d \mid 2f + 1$, so $n = 1$ if and only if $d = 2f + 1$ and $a = 1$. Thus $a \neq 1$ whenever $G = \text{Aut}(S)$. \square

Theorem 4.2. *Let $S = {}^2B_2(q^2)$, where $q^2 = 2^{2f+1}$, $f \geq 1$ and let $S \leq G \leq \text{Aut}(S)$ with $|G : S| = d$. Then the degrees of the irreducible characters of G lying over characters of S of degree $(q^2 - r + 1)(q^2 - 1)$ are precisely $(q^2 - r + 1)(q^2 - 1)b$, where b is a positive divisor of d , with the following exceptions:*

- (i) If $f \equiv 1$ or $2 \pmod{4}$, and $G = \text{Aut}(S)$, then $b \neq 1$,
- (ii) If $f \equiv 0$ or $3 \pmod{4}$, and $G = \text{Aut}(S)$, then $b \neq 3$.

Proof. Reasoning as in the proof of Theorem 4.1, we note that for $b \mid d$, there is a character of G of degree $(q^2 - r + 1)(q^2 - 1)b$ lying over a character θ of S of degree $(q^2 - r + 1)(q^2 - 1)$ if and only if

$$I_G(\theta) = S\langle\varphi^n\rangle, \quad \text{where } n = \left(\frac{2f+1}{d}\right)b.$$

By Lemma 3.3, such a character θ of S exists unless when

- (A1) $n = 1$ and $f \equiv 1$ or $2 \pmod{4}$, or
- (A2) $n = 3$ and $f \equiv 0$ or $3 \pmod{4}$.

In cases (A1) and (A2), N does not stabilize any Y_j .

Suppose that (A1) holds. Since $d \mid 2f + 1$, so $n = 1$ if and only if $d = 2f + 1$ and $b = 1$. Thus $b \neq 1$ whenever $f \equiv 1$ or $2 \pmod{4}$ and $G = \text{Aut}(S)$.

Now, suppose that (A2) holds. In this case, $n = (2f + 1)b/d = 3$ implies that either $d = 2f + 1$ and $b = 3$, or $d = (2f + 1)/3$ and $b = 1$. If $d = 2f + 1$ and $b = 3$, then $G = \text{Aut}(S)$ and exception (ii) follows. Also if $d = (2f + 1)/3$ and $b = 1$, then $G = S\langle\varphi^3\rangle$

and $I_G(\theta) = G$. By Lemma 3.3, there is a character Y_j with $j = (q^2 + r + 1)/5$ invariant under φ , hence $I_G(Y_j) = G$. Thus G has a character of degree $(q^2 + r + 1)(q^2 - 1)b$ with $b = 1$. The proof is now complete. \square

By a similar argument as above, we obtain the following result on characters of S of degree $(q^2 + r + 1)(q^2 - 1)$.

Theorem 4.3. *Let $S = {}^2B_2(q^2)$, where $q^2 = 2^{2f+1}$, $f \geq 1$ and let $S \leq G \leq \text{Aut}(S)$ with $|G : S| = d$. Then the degrees of the irreducible characters of G lying over characters of S of degree $(q^2 + r + 1)(q^2 - 1)$ are precisely $(q^2 + r + 1)(q^2 - 1)c$, where c is a positive divisor of d , with the following exceptions:*

- (i) *If $f \equiv 0$ or $3 \pmod{4}$, and $G = \text{Aut}(S)$, then $c \neq 1$,*
- (ii) *If $f \equiv 1$ or $2 \pmod{4}$, and $G = \text{Aut}(S)$, then $c \neq 3$.*

Finally, note that Theorem A follows immediately from Lemma 2.2 and Theorems 4.1-4.3.

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